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L^2 boundedness of the solutions to the 2D Navier-Stokes equations and hyperbolic Navier-Stokes equations *

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1 Introduction

In the case of the Cauchy problem of the linear heat equations

$$u_t - \Delta u = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{on} \quad \mathbb{R}^2 \tag{1.2}$$

we see that the solutions for the initial data $u_0 \in L^1(\mathbb{R}^2)$ satisfy

$$\lim_{t\to\infty} t\|u(t)\|_{L^2}^2 = \frac{1}{8\pi}\left|\int_{\mathbb{R}^2} u_0(x)\,dx\right|^2.$$

(For the proof see [9]). Thus, we can observe that the solution $u = u(t,x)$ to the Cauchy problems for the linear heat equations (1.1) and (1.2) does not have the $L^2((0,\infty)\times\mathbb{R}^2)$-boundedness for the initial data u_0 in $L^1(\mathbb{R}^2)$, in general. In the case of the Cauchy problems of the linear heat equations and also the linear damped wave equations, if we choose the initial data u_0 to be in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ (see the definition below) in stead of the $L^1(\mathbb{R}^2)$, then we can show the $L^2((0,\infty)\times\mathbb{R}^2))$-boundedness of the solutions (cf. [9], [14], [19]).

For the Cauchy problem of the Navier-Stokes equations, Leray [12], Hopf [7] showed the existence of weak solutions, and Masuda [13] showed that the $L^2(\mathbb{R}^2)$-norm of weak

*This article is baced on the a joint work with Masashi Misawa (Kumamoto University) and Taku Yanagisawa (Nara Women's University) solutions tends to zero as time goes to infinity. Wiegner [20] showed the decay rate of the weak solutions, for instance, $\|u(t)\|_{L^2} = O(t^{-\frac{1}{2}})$ as $t \to \infty$ when the initial data $u_0 \in L^1(\mathbb{R}^2)$. In [15] and [16], Miyakawa considered the Cauchy problem for the Stokes equations and the Navier-Stokes equations and proved that $\|\nabla u(t)\|_{\mathcal{H}^1} = O(t^{-\frac{1}{2}})$ as $t \to \infty$ for the solutions in the case that the initial data $u_0 \in \mathcal{H}^1(\mathbb{R}^2)$.

In this article, we will report that the solution to the Cauchy problems of the Navier-Stokes equations and the 2D Hyperbolic Navier-Stokes equations, for the initial data in $L^1(\mathbb{R}^2)^2$ and in the natural energy class, has the $L^2((0,\infty)\times\mathbb{R}^2)$-boundedness. In order to show these facts, the key points are the divergence free condition $\nabla\cdot u = 0$ and the nonlinear term's structure for the Navier-Stokes equations.

We consider the 2D Navier-Stokes equations

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi = 2\nabla\cdot S & \text{in } (0,\infty)\times\mathbb{R}^2, \\ \nabla\cdot u = 0 & \text{in } (0,\infty)\times\mathbb{R}^2, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^2 \end{cases} \tag{1.3}$$

where $u(t,x) = (u_1(t,x), u_2(t,x))$ and $\pi(t,x)$ denote unknown velocity field and scalar pressure, $u_0(x)$ is given vector function, and S is the deformation tensor given by

$$S = \frac{\mu}{2}\left((\nabla u) + {}^T(\nabla u)\right). \tag{1.4}$$

In this situation the divergence free condition $\nabla\cdot u = 0$ implies that

$$2\nabla\cdot S = \mu\Delta u.$$

We replace the Fourier type law (1.4) by the law of Cattaneo type relation

$$(1+\tau\partial_t)S = \frac{\mu}{2}\left((\nabla u) + {}^T(\nabla u)\right) \tag{1.5}$$

for small $\tau > 0$, which represents the first order Taylor approximation of the delayed deformation condition

$$\begin{aligned} S(t+\tau,x) &= S(t,x) + \tau\partial_t S(t,x) + \cdots \\ &= \frac{\mu}{2}\left((\nabla u) + {}^T(\nabla u)\right). \end{aligned}$$

Applying $\tau\partial_t$ to $(1)_1$ and adding the resulting equation to the original one gives us in view of (1.5) that

$$\begin{cases} \tau\partial_t^2 u - \mu\Delta u + \partial_t u + (1+\tau\partial_t)\nabla\pi = -(1+\tau\partial_t)((u\cdot\nabla)u), \\ \nabla\cdot u = 0, \\ u(0,x) = u_0, u_t(0,x) = u_1. \end{cases} \tag{1.6}$$

This hyperbolic fluid model (1.6) was already derived in [2] and [3].

Here, we denote the projection P with respect to the Helmholtz decomposition in $\mathbb{R}^2$ by

$$Pu = u + \nabla\pi, \quad -\Delta\pi = \nabla\cdot u.$$

Then, the projection P is a bounded operator from $L^2(\mathbb{R}^2)^2$ to $L^2_\sigma(\mathbb{R}^2)$ where $L^2(\mathbb{R}^2)$ is the standard L^2 space and

$$L^2_\sigma(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2)^2 \,:\, \nabla \cdot u = 0\}.$$

Applying P to (1.6), we have the Hyperbolic Navier-Stokes equations

$$\begin{cases} \tau\partial_t^2 u - \mu\Delta u + \partial_t u = -P(1+\tau\partial_t)((u\cdot\nabla)u), \\ u(0) = u_0, u_t(0) = u_1. \end{cases} \tag{1.7}$$

Before stating our main results, we shall introduce the function spaces. We use the standard Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ and the usual Lebesgue space $L^p(\mathbb{R}^n) = W^{0,p}(\mathbb{R}^n)$, $(1 \le p \le \infty)$ with the norm $\|\cdot\|_{W^{m,p}}$ and $\|\cdot\|_{L^p}$, respectively. For simplicity, we shall use the notation $H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$ with the norm $\|\cdot\|_{H^m}$.

R. Racke and J. Saal [10, 11] proved the following local and global in time existence theorem to the Hyperbolic Navier-Stokes equations (1.7) in $\mathbb{R}^n$ $(n \ge 2)$.

Theorem 1. *([10]) Let $n \ge 2$ and $m > \frac{n}{2}$. For each*

$$(u_0, u_1) \in (H^{m+2}(\mathbb{R}^n) \times H^{m+1}(\mathbb{R}^n)) \cap L^2_\sigma(\mathbb{R}^n)$$

there exists a time $T > 0$ and a unique solution (u, π) to the equations (1.7) satisfying

$$\begin{aligned} &u \in C^2([0,T], H^m(\mathbb{R}^n)) \cap C^1([0,T], H^{m+1}(\mathbb{R}^n)) \\ &\qquad \cap C^0([0,T], H^{m+2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)), \\ &\nabla(p + \tau p_t) \in C^0([0,T], H^m(\mathbb{R}^n)). \end{aligned}$$

The existence time T can be estimated from below as

$$T > \frac{1}{1 + C(\|u_0\|_{H^{m+2}} + \|u_1\|_{H^{m+1}})}$$

with a constant $C > 0$ depending only on m and the dimension n.

Theorem 2. *([11]) Let $m_1 \ge 3, m \ge m_1 + 9, 4 < q < \infty, 1/q + 1/p = 1$. There exists $\varepsilon > 0$ such that if*

$$\|(u_0, u_1)\|_{H^{m+2}\times H^{m+1}} + \|(u_0, u_1)\|_{L^1} + \|(u_0, u_1)\|_{W^{m_1+6,p}\times W^{m_1+5,p}} < \varepsilon,$$

then there exists a unique global solution (u, π) to the hyperbolic Navier-Stokes equations (1.7), satisfying

$$\begin{aligned} &u \in C^2([0,T], H^m(\mathbb{R}^n)) \cap C^1([0,T], H^{m+1}(\mathbb{R}^n)) \\ &\qquad \cap C^0([0,T], H^{m+2}(\mathbb{R}^n)), \\ &\nabla(p + \tau p_t) \in C^0([0,T], H^m(\mathbb{R}^n)). \end{aligned}$$

Also, there is $M_0 > 0$, independent of T such that

$$M(T) \le M_0$$

where

$$\begin{aligned} M(T) &= \sup_{0\le t\le T} \Big\{(1+t)^{1-\frac{2}{q}}\|u(t)\|_{W^{m_1,q}} + (1+t)^{\frac{3}{2}-\frac{2}{q}}\|(u_t(t),\nabla u(t))\|_{W^{m_1,q}} \\ &\quad +(1+t)^{\frac{1}{2}}\|u(t)\|_{H^m} + (1+t)\|(u_t(t),\nabla u(t))\|_{H^m}\Big\}. \end{aligned}$$

Remark 1.1. *From Theorem 2, we see that for $t > 0$*

$$\|u(t)\|_{L^2} \le C(1+t)^{-1/2},$$

$$\|(\partial_t u(t),\nabla u(t))\|_{L^2} \le C(1+t)^{-1}$$

where $C > 0$ is independent of t.

Our main result is the following.

Theorem 3. *Let $n = 2$. The assumptions of Theorem 1 and 2 hold. Then, the solutions $u(t)$ to the hyperbolic Navier-Stokes equations (1.7) satisfiy the following property*

$$\int_0^t \|u(s)\|_{L^2}^2 ds < C$$

where C is independent of t.

Note that we have the same results to the Cauchy problem of the Navier-Stokes equations (1.3) and (1.4) in $\mathbb{R}^2$ for large initial data in $L^1(\mathbb{R}^2)^2 \cap L^2_\sigma(\mathbb{R}^2)$.

2. Key Lemmas.

We will start with the definitions of function spaces (refer to [5]).

Definition 1. (Hardy space) *Let $n \ge 2$. The Hardy space consists of functions f in $L^1(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sup_{r>0} |\phi_r * f(x)| dx$$

is finite, where $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ for $r>0$ and ϕ is a smooth function on $\mathbb{R}^n$ with compact support in an unit ball with center of the origin $B_1(0) = \{x \in \mathbb{R}^n; |x|<1\}$.

The definition dose not depend on choice of a function ϕ.

Definition 2. (functions of bounded mean oscillation) *Let $n \geq 2$ and f be a locally integrable in $\mathbb{R}^n$, that is $f \in L^1_{loc}(\mathbb{R}^n)$. We say that f is of bounded mean oscillation (abbreviated as BMO) if*

$$\|f\|_{BMO} = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f - (f)_B| dx < \infty,$$

where the supremum ranges over all finite ball $B \subset \mathbb{R}^n$, $|B|$ is the n-dimensional Lebesgue measure of B, and $(f)_B$ denotes the integral mean of f over B, namely $(f)_B = \frac{1}{|B|} \int_B f(x)dx$.

The class of functions of BMO, modulo constants, is a Banach space with the norm $\|\cdot\|_{BMO}$ defined above.

We will prepare the decisive Fefferman-Stein inequality, which means the duality between $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, $(\mathcal{H}^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$. For the proof, see [5].

Lemma 2.1. (Fefferman-Stein inequality) *Let $n \geq 2$. There is a positive constant C depending only on n such that if $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, then*

$$\left| \int_{\mathbb{R}^n} fg dx \right| \leq C \|f\|_{\mathcal{H}^1} \|g\|_{BMO}.$$

Also, we shall use the following Poincaré inequality in $\mathbb{R}^2$, which is proved by the definition of BMO and the usual Poincaré inequality in $\mathbb{R}^2$. For the detail of the proof, see [14] etc.

Lemma 2.2. (Poincaré inequality) *For $f \in H^1(\mathbb{R}^2)$, the following inequality holds.*

$$\|f\|_{BMO} \leq C \|\nabla f\|_{L^2}. \tag{2.1}$$

Here, we introduce the function space $W_0^{1,p}(\mathbb{R}^n), (1 < p < \infty, n \geq 2)$ by

$$W_0^{1,p}(\mathbb{R}^n) = \left\{ u : \frac{u}{w(x)} \in L^p(\mathbb{R}^n), \nabla u \in L^p(\mathbb{R}^n) \right\}$$

where $w(x) = 1 + |x|$ if $p \neq n$, and $w(x) = (1 + |x|) \log(2 + |x|)$ if $p = n$. The following Lemma proved by Amrouche and Nguyen [1] is key Lemma to show the linear parts in our main results of this article.

Lemma 2.3. ([*1*])*Let $n \geq 2$. If $f \in L^1(\mathbb{R}^n)$ and $\nabla \cdot f = 0$, then $\int_{\mathbb{R}^n} f(x)\, dx = 0$ and*

$$\left| \int_{\mathbb{R}^n} fg dx \right| \leq C \|f\|_{L^1} \|\nabla g\|_{L^n}$$

for $g \in W_0^{1,n}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

In order to estimate the nonlinear terms, we shall use the Lemmas 2.1 and 2.2, and also use the following key Lemma, which is concerned with the property of the nonlinear term's structure for the Navier-Stokes equations.

Lemma 2.4. *([4]) If $\nabla \cdot u = 0$, then*

$$\|(u \cdot \nabla) u\|_{\mathcal{H}^1} \leq C \|u\|_{L^2} \|\nabla u\|_{L^2}.$$

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